10.1 Sequences and Series

a **sequence** is a list of numbers in a particular order

for example, $\{1, 2, 3, 4, 5\}$ first five natural numbers

finite sequence because the sequence ends

$\{1, 2, 3, 4, 5, \ldots\}$ all natural numbers

infinite sequence

$\{2, 4, 6, 8, 10, \ldots\}$ even numbers

we can also list using explicit formula

"a" is the name of the sequence

$$\{1, 2, 3, 4, 5, \ldots\} = \{a_n\}_{n=1}^{\infty}$$

∞ → where we end

$n=1$ → where do we start counting?

$a_1 \quad a_2 \quad a_3$

n^{th} term

$$= \{n\}_{n=1}^{\infty}$$

$$\{2, 4, 6, 8, 10, \ldots\} = \{2n\}_{n=1}^{\infty}$$

Sometimes we use the <u>recurrence relation</u>

$\{1, 2, 3, 4, 5, \ldots\}$ each is one more than the one before

$a_1 = 1, \quad a_{n+1} = a_n + 1$

for example, $a_2 = a_1 + 1 = 1 + 1 = 2$

$a_3 = a_2 + 1 = 2 + 1 = 3$

$\vdots$

$\{2, 4, 6, 8, 10, \ldots\}$

$a_1 = 2, \quad a_{n+1} = a_n + 2$

$\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$ Fibonacci Sequence

each term is sum of previous two

$a_1 = 1, \quad a_2 = 1, \quad a_{n+2} = a_{n+1} + a_n$

e.g. $a_3 = a_2 + a_1$

a <u>sequence</u> is said to <u>converge</u> if $\lim_{n\to\infty} a_n$ exists

if $\lim_{n\to\infty} a_n$ DNE, then sequence <u>diverges</u>

<u>example</u> $a_n = \dfrac{(-1)^n n}{2n^2+1}$ $n=1,2,3,\ldots$

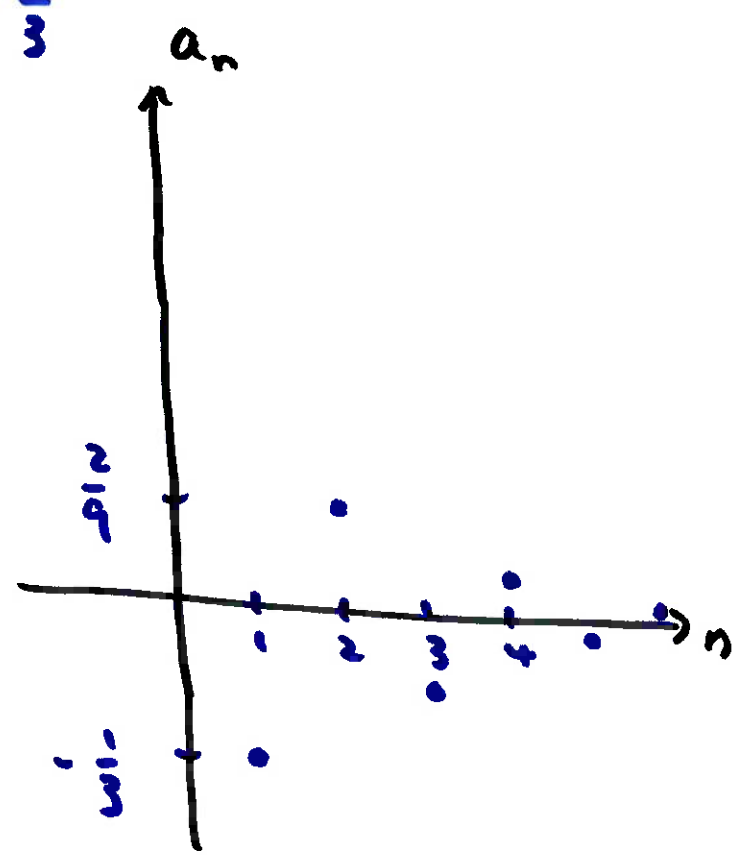
first few: $a_1 = \dfrac{(-1)^1 \cdot 1}{2(1)^2+1} = -\dfrac{1}{3}$

$a_2 = \dfrac{2}{9}$

$a_3 = \cancel{\dfrac{3}{19}}\ -\dfrac{3}{19}$

$a_4 = \dfrac{4}{33}$

$a_5 = \dfrac{-5}{51}$



the magnitude appears to decrease as n increases

$$a_n = \frac{(-1)^n n}{2n^2+1}$$

$$\lim_{n\to\infty} \frac{(-1)^n n}{2n^2+1} = \underbrace{\lim_{n\to\infty} (-1)^n}_{\text{no effect on magnitude}} \cdot \underbrace{\lim_{n\to\infty} \frac{n}{2n^2+1}}_{\text{magnitude here, goes to 0}} = 0$$

Since $\lim_{n\to\infty} a_n$ exists, this sequence <u>converges</u> (or is <u>convergent</u>)

what about

$$a_n = \frac{2n}{n+1} \qquad n = 1, 2, 3, \ldots$$

converges?

$$\lim_{n\to\infty} \frac{2n}{n+1} = 2$$ since limit exists, sequence converges

limits (review)

$$\lim_{n\to\infty} \frac{2n}{n+1}$$ when n is large, $n+1 \approx n$ (e.g. $n = 100{,}000$)

$$\frac{2n}{n+1} \approx \frac{2n}{n} = 2$$

or, we can use **l'Hospital's Rule**

when limit $\to \frac{\infty}{\infty}$ or $\frac{0}{0}$, then $\lim_{n\to\infty} \frac{f(x)}{g(x)} = \lim_{n\to\infty} \frac{f'(x)}{g'(x)}$

$$\lim_{n\to\infty} \frac{n}{2n^2+1} \to \frac{\infty}{\infty}$$

by l'Hospital's Rule

deriv. of n

$$= \lim_{n\to\infty} \frac{1}{4n} = 0$$

deriv. of $2n^2+1$

a **series** is the sum of the terms in a sequence

$\{1, 2, 3, 4, 5, \ldots\}$ sequence

$1+2+3+4+5+\ldots$ ~~series~~ series

like with sequences, we can express series compactly

$$1+2+3+4+5+\ldots = \sum_{n=1}^{\infty} n$$

end at ∞

sigma means sum

because each term is equal to n

start adding at $n=1$

$n=1$ $n=2$ $n=3$

$$2+4+6+8+\ldots = \sum_{n=1}^{\infty} 2n$$

$n=1$ $n=2$

(so $a_n = 2n$)

infinite series → no end to adding

there is an end

$$1+2+3 = \sum_{n=1}^{3} n$$

finite series

example $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$

$$= \underset{n=1}{\frac{1}{2^1}} + \underset{n=2}{\frac{1}{2^2}} + \underset{n=3}{\frac{1}{2^3}} + \frac{1}{2^4} + \frac{1}{2^5} + \cdots = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

the sum of the first n terms is called the <u>n^{th} partial sum</u>, S_n

first partial sum $S_1 = \frac{1}{2}$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$$

$$\vdots$$

$$S_{10} = \cdots = \frac{1023}{1024}$$

note the partial sums appear to approach $1 \rightarrow \lim_{n\to\infty} S_n = 1$

if the partial sums appear to approach some finite number,

we say the series converges (or is convergent)

otherwise, the series diverges (or is divergent)

$\Rightarrow$ does $\lim_{n\to\infty} S_n$ exist?

yes: converges

no: diverges

so, $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges

$\sum_{n=1}^{\infty} n$ diverges because $1+2+3+4+\ldots$ does not have a limit

<u>example</u> $\displaystyle\sum_{n=1}^{\infty} \cos(n\pi)$

$$= \underset{n=1}{\cos(\pi)} + \underset{n=2}{\cos(2\pi)} + \underset{n=3}{\cos(3\pi)} + \underset{n=4}{\cos(4\pi)} + \cdots$$

$$= -1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 \ldots$$

partial sums

$S_1 = -1$

$S_2 = 0$

$S_3 = -1$

$S_4 = 0$

$S_5 = -1$

$\vdots$

do these appear to settle down around some finite ~~unit~~ number?

$\lim_{n\to\infty} S_n$ exists? NO! this series <u>DIVERGES</u>

$\{-1, 0, -1, 0, -1, \ldots\}$
sequence of partial sums